Viscosity Solution of Bellman-Isaacs Equation Arising in Non-linear Uncertain Object Control*

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Abstract: The problem of optimal control for a class of non-linear objects with uncontrolled bounded disturbances is formulated in the sense of a differential game. In case of problems with quadratic quality functional, the problem of optimal control search is reduced to finding of solution of Hamilton-Jacobi-Isaacs equation. Solutions of this equation at the rate of functioning of the object are searched by means of special algorithmic procedures obtained with the use of viscosity solution theory. The obtained results may be used for solving of theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, engineering, control and navigation.

Keywords: Non-linear systems, differential games, Hamilton-Jacobi-Isaacs equation, viscosity solution

1. INTRODUCTION

Successful implementation of obtained theoretical results in a number of problems is connected with solving of partial first-order derivative equations. Such partial derivative equations appear under solving of a great number of theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, engineering, control, etc. Such equations are Hamilton-Jacobi equation in theoretical mechanics (Arnold, 1977), Bellman equation in theory of optimal control (Bellman, 1957), Isaacs equation (Isaacs, 1965), eikonal equation in geometrical optics (Courant, 1961), Burgers and Hopf limit equations in gas dynamics and hydrodynamics (Bardi M., 1997; Crandal M. G., 1992), etc.
The method of characteristics proposed in the first half of the 19^{th} century by O. Cauchy for solving boundary problems for such equations reduces integrating of partial first-order derivative equations to integrating of a system of ordinary differential equations. This method is based on the fact that invariance of graph of the classical solution for a boundary problem is relative to the characteristics. However, in case of partial derivative nonlinear equation, smooth solution exists only locally.
In 1950-1970s a lot of mathematicians paid much attention to generalized solutions of Hamilton-Jacobi and other types of equations (Evans L.C. 1998; Bardi M., 1998). Developed methods mainly based on integral methods and integral properties of generalized solutions.
In early 1980s a concept of viscosity solution was introduced the existence of which was proved by method of disappearing viscosity (Crandall and all, 1992). The method is also being developed at present time. The researches pay attention to analytical, constructive and numerical methods of construction of viscosity solutions (Cacace and all, 2011) and application of theoretical results to solving of various applied problems.Another well-known concept of the generalized solution based on idempotent analysis was proposed in works by V.P. Maslov and his disciples (1992). By means of this approach linearizing convex problems, Hamilton-Jacobi equations with a convex Hamiltonian and their applications to problems of mathematical physics are studied.

Optimal control problems and differential games are connected one way or another with a search for solutions of Hamilton-Jacobi-Bellman, Isaaks equations. To solve such equations, constructive and numerical methods (including grid ones) were developed (Subbotin and all, 1993, 1994). An important result of the theory of minimax solutions of first-order PDE being a base for differential game theory is proving the equivalence of concepts of minimax and viscosity solutions (Subbotin, 1995).

Within the frameworks of minimax solution concept originating from the theory of position differential games (Krasovsky, Subbotin, 1988) developed by school of N. N. Krasovsky on the base of minimax evaluations and operations, theorems of existence and uniqueness, correctness and content-richness of minimax solution concept for various types of boundary problems of partial first-order PDE were proved.

Despite available theoretical results in this area, the issue of Hamilton-Jacobi-Isaacs equation solution in the problems of differential games with non-linear indefinite dynamic objects in the rate of their functioning persists and is important today.

2. NON-LINEAR OPTIMAL REGULATER

2.1. *Problem statement*

Consider a dynamical non-linear uncertain system described by the ordinary differential equation

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$$\frac{d}{dt}x(t) = f(x) + g_1(x)w(t) + g_2(x)u(t),\ x(t_0) = x_0, \quad (2.1)$$
$$y(t) = Hx(t).$$

Here $x(\cdot) = \{x(t) \in R^n, t \in [t_0, T]\}$ is a state vector of the system; $x(\cdot) \in \Omega_x$, $X_0 \in \Omega_x$ is a range of possible initial conditions of the system; $y \in R^m$, $m \le n$ is an output of the system; $u \in R^r$ is a control; $w \in R^k$ is a disturbance; $f(x),\ g_1(x),\ g_2(x)$ are continuous matrix functions.

It is assumed that for all x system (2.1) is controllable and observable, $t \in R^+$. In addition, assume that functions $f(x),\ g_1(x),\ g_2(x)$ are smooth enough (C_∞), so that for any $(t_0, x_0) \in R_+ \times \Omega_x$ only one solution $x(t, t_0, x_0)$ of (2.1) equation is possible and the corresponding output of the system $y(t) = Cx(t, x_0)$ is unique.

Assumption 2.1. The vector function $f(x)$ is continuous differentiable with respect to $x \in \Omega_x$, i.e. $f(\cdot) \in C^1(\Omega_x)$ and $g_1(\cdot),\ g_2(\cdot) \in C^0(\Omega_x)$.

Assumption 2.2. Without loss of generality, assume that condition $x = 0 \in \Omega_x$ is a point of equilibrium of the system under $u = 0,\ w = 0$, so that $f(0) = 0$ and $g_1(x) \ne 0$, $g_2(x) \ne 0,\ \forall x \in \Omega_x$.

While considering disturbance $w(t)$ as an action of some player against successful performance of a control problem, we state the control problem in the sense of a differential game of two players: G_u and G_w. Controls $u(t) \in U$ and $w(t) \in W$ will be organized using the state feedback principle.

So, in the present section, the problem of control of non-linear uncertain object (2.1) will be considered in the sense of the minimax theory.

Introduce the cost functional of the differential game

$$J(x,u,w) = K(x(T)) + \frac{1}{2}\int_{t_0}^{T} \left\{ y^{\mathrm{T}}(t)Q\,y(t) + u^{\mathrm{T}}(t)Ru(t) - w^{\mathrm{T}}(t)Pw(t) \right\} dt. \quad (2.2)$$

In functional (2.2) a symmetrical matrix Q is at least positively semidefinite, P and R matrices are positive definite.

Assumption 2.3. Limits on control actions U and W, where the task is executed successfully differential game, determined by the respective values of the matrices R, P, parameters σ_i, $i = 1,...k$ and matrix $g_1(x),\ g_2(x)$.

Let element $\xi = (x(t), u(t), w(t))$ be a permissible controllable process. Functions of class $x(\cdot) \in C^1([t_0, T], R^n)$, $u(\cdot) \in C^1([t_0, T], R^r)$, $w(\cdot) \in C^1([t_0, T], R^k)$ will be considered as permissible elements $\xi = (x(t), u(t), w(t))$.

The problem of differential game consists in construction of an optimal strategy with feedback for players G_u and G_w, i.e. in finding of control $u(t)$ minimizing a functional of (2.2) on the object (2.1) under corresponding counteraction to control $w(t)$.

2.2. Optimal controls of differential game

Make two assumptions:

Assumption 2.4. Let $f(x),\ g_1(x),\ g_2(x)$ be smooth enough functions, so that function $V(t,x)$ determined as

$$V(t,x) \triangleq \inf_{u\in U}\ \sup_{w\in W}\ J(x,u,w) \quad (2.3)$$

is a differentiable function under any permissible strategies of players $G_w,\ G_u \in L_2(0,\infty)$.

Assumption 2.5. A function $V(t,x)$ determined in (2.3) is locally Lipschitz in Ω_x.

In general case, value of an assigned function $V(t,x)$ is a solution of dynamic programming problem connected with partial differential equation of the first order (first order PDE) Hamilton-Jacobi-Isaacs (Issacs, 1965).

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_w H\left\{x, u, w, \frac{\partial V(t,x)}{\partial x}\right\} = 0, \quad (2.4)$$
$$V(T, x(T)) = K(x(T)),$$

where H is Hamiltonian

$$H\left\{x, u, w, \frac{\partial V(t,x)}{\partial x}\right\} = $$
$$= \frac{1}{2}\left\{y^{\mathrm{T}}(t)Q\,y(t) + u^{\mathrm{T}}(t)Ru(t) - w^{\mathrm{T}}(t)Pw(t)\right\} + \quad (2.5)$$
$$+ \frac{\partial V(t,x)}{\partial x}\left\{f(x) + g_1(x)w(t) + g_2(x)u(t)\right\}.$$

Optimum controls $u(t)$ and $w(t)$ when performing Assumptions 2.3, defined by the relations

$$w(t) = P^{-1}g_1^{\mathrm{T}}(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}},$$
$$u(t) = -R^{-1}g_2^{\mathrm{T}}(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}}, \quad (2.6)$$

where vector $\partial V(x)/\partial x$ is determined by solution of Hamilton-Jacobi-Isaacs equation:

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}f(x) - \quad (2.7)$$
$$-\frac{1}{2}\frac{\partial V(t,x)}{\partial x}\Pi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}} + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) = 0,$$
$$V(T, x(T)) = K(x(T)),$$

where

$$\varPi(x) = g_2(x)R^{-1}g_2^{\mathrm{T}}(x) - g_1(x)P^{-1}g_1^{\mathrm{T}}(x)\,. \tag{2.8}$$

The main difficulty under implementation of controls in form (2.6) consists in finding of vector $\partial V(x) / \partial x(t)$ satisfying scalar partial derivative equation (2.7).

2.3. Conditions of existence of optimal solution

Conditions of existence of optimal solution of the set problem are determined by properties of matrix $\varPi(x)$. To determine properties of this matrix, consider in this section problem of synthesis of stabilizing controls for system (2.1), i.e. consider the problem with unlimited time of transition process. Quality functional for such a problem has the form

$$J(x,u,w) = \tag{2.9}$$

$$= \lim_{T\to\infty} \frac{1}{2}\int_0^{\mathrm{T}} \left\{ y^{\mathrm{T}}(t)Q\,y(t) + u^{\mathrm{T}}(t)R\,u(t) - w^{\mathrm{T}}(t)Pw(t) \right\} dt.$$

Show that system (2.1) with controls (2.6) where vector $\partial V(x) / \partial x$ is determined by solution of Hamilton-Jacobi-Isaacs:

$$\frac{\partial V(x)}{\partial x} f(x) - \frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) = 0 \tag{2.10}$$

is asymptotically stable, i.e. $\lim_{t\to\infty} x(t) = 0$ under fulfillment of certain requirements to matrix $\varPi(x)$.

Assumption 2.6. Let matrix $\varPi(x)$ be at least positively semidefinite.

The system (2.1) with controls (2.6) is determined by the expression

$$\frac{d}{dt}x(t) = f(x) - \varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}},\ x(t_0) = x_0, \tag{2.11}$$

$$y(t) = C(x).$$

Theorem 2.1. System (2.11) is asymptotically stable if and only if

$$\frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} \geq \frac{\partial V(x)}{\partial x} f(t,x),\ \forall x \neq 0.$$

where $\varPi(x)$ is at least positively semidefinite matrix.

Proof. From equation (2.10) we have

$$\frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} = \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}x^{\mathrm{T}}H^{\mathrm{T}}QHx(t)\,.$$

Hence, as soon as $x^{\mathrm{T}}H^{\mathrm{T}}QHx(t) \geq 0$,

$$\frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} > \frac{\partial V(x)}{\partial x} f(x)\,. \tag{2.12}$$

Define requirements to assignment of matrices P and R under which Assumption 2.3 is true, i.e. matrix $\varPi(x)$ is at least positively semidefinite.

In accordance with Lyapunov theorem, system (2.11) is asymptotically stable if the following condition is fulfilled:

$$\frac{\partial V(x)}{\partial x}\left[f(x) - \varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} \right] \leq -\omega_3(|x|),\ \omega_3(|x|) \geq 0\,.$$

Assign $\omega_3(|x|)$ as $\omega_3(|x|) = \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)$ where Q is a positively semidefinite matrix of quality functional (2.9). Then, taking into account (2.11) and (2.12), we have

$$\frac{\partial V(x)}{\partial x} f(x) - \frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) - \frac{1}{2}\frac{\partial V(x)}{\partial x}\varPi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}} \leq 0.$$

In view of (2.11), we get that the matrix $\varPi(x)$ is at least positively semidefinite. As it is seen from (2.8), this property of matrix $\varPi(x)$ may be ensured (under known matrices $g_1(x)$ and $g_2(x)$) by appropriate assignment of matrices R and P.

3. VISCOSITY SOLUTION OF EQUATION (2.8)

Development of convex and uneven analysis in 1970s allowed application of new results and methods based on generalizations of differentiability concept to study of generalized solutions of a first order PDE. In early 1980s M. Crandall and P. L. Lions introduced concept of viscosity solution (Crandall and all, 1992). Introduce one of equivalent definitions of viscosity solution.

Definition 3.1. Upper viscosity solution of equation

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x) + \frac{1}{2}x^{\mathrm{T}}H^{\mathrm{T}}QHx(t) - \frac{1}{2}\frac{\partial V(t,x)}{\partial x}\varPi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}} = 0 \tag{3.1}$$

is continuous function $\varphi^{\mathrm{T}}(t)x(t)$ meeting the following condition: if difference of functions $V(t,x) - \varphi^{\mathrm{T}}(t)x(t)$ reaches a local minimum in point $(t^*, x^*) \in \Omega$ and function $\varphi^{\mathrm{T}}(t)x(t)$ is differentiable in this point, the following in inequality is true:

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x) + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t - \frac{1}{2}\frac{\partial V(t,x)}{\partial x}\varPi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}} \leq 0. \tag{3.2}$$

Definition 3.2. Lower viscosity solution of equation (3.1) is continuous function $\varphi^{\mathrm{T}}(t)x(t)$ meeting the following condition: if difference of functions $V(t,x) - \varphi^{\mathrm{T}}(t)x(t)$ reaches a local maximum in point $(t^*, x^*) \in \Omega$ and function $\varphi^{\mathrm{T}}(t)x(t)$ is differentiable in this point, the following inequality is true:

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(x) + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) -$$

$$-\frac{1}{2}\frac{\partial V(t,x)}{\partial x}\Pi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}} \geq 0. \tag{3.3}$$

Definition 3.3. Viscosity solution is a function which is simultaneously upper and lower solution, i.e. the following condition is true:

$$V(t,x) = \varphi^{\mathrm{T}}(t)x(t). \tag{3.4}$$

On the base of this definition, find equation for function $\varphi(t)$. Write total derivative of function $V(t,x)$:

$$\frac{dV(t,x)}{dt} = \left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}} x(t) + \\ +\varphi^{\mathrm{T}}(t)f(x) - \varphi^{\mathrm{T}}(t)\Pi(x)\left\{\frac{\partial V(t,x)}{\partial t}\right\}^{\mathrm{T}}. \tag{3.5}$$

On the other side, taking into account that

$$\frac{dV(t,x)}{dt} = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}\frac{dx(t)}{dt}$$

and in view of (2.11) and (3.1), i.e.

$$\frac{\partial V(t,x)}{\partial t} = \\ = -\frac{\partial V(t,x)}{\partial x}f(x) - \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) + \frac{1}{2}\frac{\partial V(t,x)}{\partial x}\Pi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}}.$$

we have:

$$\frac{dV(t,x)}{dt} = \\ = -\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) - \frac{1}{2}\frac{\partial V(t,x)}{\partial x}\Pi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}}. \tag{3.6}$$

Making (3.5) and (3.6) equal, we get

$$\left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}} x(t) + \varphi^{\mathrm{T}}(t)f(x) - \varphi^{\mathrm{T}}(t)\Pi(x)\left\{\frac{\partial V(t,x)}{\partial t}\right\}^{\mathrm{T}} = \\ = -\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) - \frac{1}{2}\frac{\partial V(t,x)}{\partial x}\Pi(x)\left\{\frac{\partial V(t,x)}{\partial x}\right\}^{\mathrm{T}}$$

hence

$$\left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}} x(t) + \varphi^{\mathrm{T}}(t)f(x) + \\ +\left[\frac{1}{2}\frac{\partial V(t,x)}{\partial x} - \varphi^{\mathrm{T}}(t)\right]\Pi(x)\left\{\frac{dV(t,x)}{dt}\right\}^{\mathrm{T}} + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) = 0.$$

Rewrite this expression taking into account that $V(t,x) = \varphi^{\mathrm{T}}(t)x(t)$ and $\partial V(t,x)/\partial x = \varphi^{\mathrm{T}}(t)$

$$\left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}} x(t) + \varphi^{\mathrm{T}}(t)f(x) - \frac{1}{2}\varphi^{\mathrm{T}}(t)\Pi(x)\varphi(t) + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) = 0, \\ \varphi^{\mathrm{T}}(T)x(T) = K(x(T)). \tag{3.7}$$

System with controls

$$u(t) = -R^{-1}g_2^{\mathrm{T}}(x)\varphi(t),\ w(t) = P^{-1}g_1^{\mathrm{T}}(x)\varphi(t) \tag{3.8}$$

has the form:

$$\frac{d}{dt}x(t) = f(x) - \Pi(x)\varphi(t),\ x(t_0) = x_0, \\ y(t) = C(x). \tag{3.9}$$

To find values of quality functional when system is described by expression (3.9), add to integrant of functional (2.2) expression $d\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}/dt$ compensating this addition outside of the integral $\varphi^{\mathrm{T}}(t_0)x(t_0) - \varphi^{\mathrm{T}}(T)x(T)$. It is not difficult to get that

$$J^0(x,\varphi) = K(x(T)) + \frac{1}{2}\varphi^{\mathrm{T}}(t_0)x(t_0) - \frac{1}{2}\varphi^{\mathrm{T}}(T)x(T).$$

As soon as $V(T,x(T)) = K(x(T))$, defining $K(x(T))$ as $K(x(T)) = 0,5\varphi^{\mathrm{T}}(T)x(T)$ we get value of quality functional under optimal controls (3.8)

$$J^0(x,\varphi) = 0,5\varphi^{\mathrm{T}}(t_0)x(t_0). \tag{3.10}$$

Note that equation (3.7) defines dynamic accordance of function $\varphi(t)$ to vector of system state $x(t)$. This circumstance under fulfillment of Assumption 2.1 and 2.2 and known initial condition $x(t_0)$ may be used for defining of initial conditions for equation (3.7).

Using the method of "extended linearizing" and taking into account Assumptions 2.1 and 2.2, represent the initial non-linear system (2.1) in the form of equivalent model of the system (Afanas'ev, 2015)

$$\frac{d}{dt}x(t) = A(x)x(t) + g_1(x)w(t) + g_2(x)u(t),\ x(0) = x_0, \\ y(t) = Cx(t). \tag{3.11}$$

As it is known, such a representation of the initial system is not unique in general (Cimen, 2008), but selection of a suitable model of form (3.11) is not considered in this work. It is only assumed that pairs $\langle A(x), g_1(x)\rangle$ and $\langle A(x), g_2(x)\rangle$ are controllable, and pair $\langle A(x), C\rangle$ is observable under all $x \in \Omega_x$.

Define vector function $\{\partial V(t)/\partial x\}^{\mathrm{T}}$ as

$$\{\partial V(t)/\partial x\}^{\mathrm{T}} = S(x)x(t). \tag{3.12}$$

Then, after differentiating (3.12) with respect to t, we have

$$\frac{d}{dt}\{\partial V(t)/\partial x\}^{\mathrm{T}} = \left\{\frac{d}{dt}S(x)\right\}x(t) + S(x)\left\{\frac{d}{dt}x(t)\right\} = \\ = \left\{\frac{d}{dt}S(x)\right\}x(t) + S(x)A(x)x(t) - \Pi(x)S(x)x(t), \tag{3.13}$$

where $\frac{d}{dt}S(x) = \sum_{i=1}^{n} S_{x_i}(x)x_i(t)$.

On the other side,

$$\frac{d}{dt}\{\partial V(x)/\partial x\}^{\mathrm{T}} = -\left\{\frac{\partial H}{\partial x}\right\}^{\mathrm{T}} = -C^{\mathrm{T}}QCx(t) - \\ -\left\{\left[\frac{\partial\left(A(x)x(t)\right)}{\partial x}\right]^{\mathrm{T}} + \left[\frac{\partial\left(g_1(x)w(t)\right)}{\partial x}\right]^{\mathrm{T}} + \left[\frac{\partial\left(g_2(x)u(t)\right)}{\partial x}\right]^{\mathrm{T}}\right\}\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}}, \tag{3.14}$$

From expressions (3.14) and (3.15) we have

$$\left[\frac{d}{dt}S(x) + \left\{\left[\frac{\partial\left(A(x)x(t)\right)}{\partial x}\right]^{\mathrm{T}} + \left[\frac{\partial\left(g_1(x)w(t)\right)}{\partial x}\right]^{\mathrm{T}} + \left[\frac{\partial\left(g_2(x)u(t)\right)}{\partial x}\right]^{\mathrm{T}}\right\}S(x) + \right.$$

$$+S(x)A(x)+A^{\mathrm{T}}(x)S(x)-S(x)\Pi(x)S(x)+H^{\mathrm{T}}QH=0.$$

When considering this equation under $x(t_0)=const$, we get Riccati algebraic equation

$$S(x_0)A(x_0)+A^{\mathrm{T}}(x_0)S(x_0)- \\ -S(x_0)\Pi(x_0)S(x_0)+H^{\mathrm{T}}QH=0. \tag{3.15}$$

The positive definite matrix $S(x_0)$ found by solving of this equation together with initial conditions $x(t_0)$ determine initial condition for equation (3.7), i.e.

$$\left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}} x(t)+\varphi^{\mathrm{T}}(t)f(x)-\frac{1}{2}\varphi^{\mathrm{T}}(t)\Pi(x)\varphi(t)+\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)=0,$$

$$\varphi^{\mathrm{T}}(t_0)x(t_0)=x^{\mathrm{T}}(t_0)S(x_0)x(t_0). \tag{3.16}$$

4. ALGORITHMIC METHOD FOR DESIGN OF CONTROLS

The main difficulty of implementation of controls in form (3.8) consists in finding of vector $\varphi(t)$ satisfying viscosity solution (3.7). One of possible ways of finding of control using equation (3.7) is a method based on approximation of this equation by Taylor series around the equilibrium point. However, the method based on representation of partial derivative inequality using approximation around the equilibrium point does not provide obtaining of more general solutions.

Propose a method of search based on application of method of algorithmic construction. First of all, it should be noted that equation (3.7) determines dynamic accordance of vector function $\varphi(t)$ to vector of state of the system $x(t)$, i.e. to state of the viscosity solution (3.4). In other words, function $\varphi(t)$ should transfer the systems from upper or lower viscosity solution to a state being simultaneously upper and lower solution, i.e. when the following condition is true: $V(t,x)=\varphi^{\mathrm{T}}(t)x(t)$.

Organize an algorithm providing transfer of the system into viscosity solution using Lyapunov function $V_L(x,\varphi)$

$$V_L(x,\varphi)=0,5\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}^2. \tag{4.1}$$

Total derivative of the function (4.1) is

$$\frac{d}{dt}V_L(x,\varphi)= \\ =\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}\left[\left\{\frac{d}{dt}\varphi^{\mathrm{T}}(t)\right\}x(t)+\varphi^{\mathrm{T}}(t)\left\{\frac{d}{dt}x(t)\right\}\right]= \tag{4.2}$$

$$=\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}\left[\left\{\frac{d}{dt}\varphi^{\mathrm{T}}(t)\right\}x(t)+\varphi^{\mathrm{T}}(t)A(x)x(t)-\varphi^{\mathrm{T}}(t)\Pi(x)\varphi(t)\right]\le 0.$$

Substituting expression $\varphi^{\mathrm{T}}(t)\Pi(x)\varphi(t)$ obtained from (3.16) into (4.2)

$$-\varphi^{\mathrm{T}}(t)\Pi(x)\varphi(t)= \\ =-2\left\{\frac{d\varphi(t)}{dt}\right\}^{\mathrm{T}}x(t)-2\varphi^{\mathrm{T}}(t)f(x)-x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)$$

we have

$$\frac{d}{dt}V_L(x,\varphi)= \\ =-\left[\frac{d}{dt}\varphi^{\mathrm{T}}(t)+\varphi^{\mathrm{T}}(t)A(x)+x^{\mathrm{T}}H^{\mathrm{T}}QH\right]x(t)\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}\le 0 \tag{4.3}$$

As soon as the expression within the brackets and the expression outside of the brackets may have both positive and negative values, relation (4.3) is true under the only condition

$$\left[\frac{d}{dt}\varphi^{\mathrm{T}}(t)+\varphi^{\mathrm{T}}(t)A(x)+x^{\mathrm{T}}H^{\mathrm{T}}QH\right]x(t)\left\{\varphi^{\mathrm{T}}(t)x(t)\right\}=0$$

Require that this condition be true when function $\varphi(t)$ providing fulfillment of "vanishing viscosity" condition corresponds to solution of the following differential equation

$$\frac{d}{dt}\varphi(t)+A^{\mathrm{T}}(x)\varphi(t)+H^{\mathrm{T}}QHx(t)=0, \\ \varphi(t_0)=S(x_0)x(t_0). \tag{4.4}$$

5. PROBLEM WITH UNLIMITED TRANSITION PROCESS END TIME

Note that the final value of the functional under $T=\infty$ does not have a common sense. Consider transition of the system from state $x(t_0)=x_0$ into $x=0$ within large enough *T*. Quality functional in such a problem has form (2.10). In this case, control determined by the equations (2.7), where the vector $\partial V(x)/\partial x$ is determined by solution of Hamilton-Jacobi-Isaacs (2.10):

$$\frac{\partial V(x)}{\partial x}f(x)-\frac{1}{2}\frac{\partial V(x)}{\partial x}\Pi(x)\left\{\frac{\partial V(x)}{\partial x}\right\}^{\mathrm{T}}+ \\ +\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)=0$$

Define viscosity solution for this case.

Definition 5.1. Upper viscosity solution of equation

$$\Psi(\gamma,x)= \\ =\gamma^{\mathrm{T}}(t)f(x)+\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)-\frac{1}{2}\gamma^{\mathrm{T}}(t)\Pi(x)\gamma(t)=0. \tag{5.1}$$

is continuous function $\gamma^{\mathrm{T}}(t)x(t)$ meeting the following condition: if difference of functions $V(x)-\gamma^{\mathrm{T}}(t)x(t)$ reaches a local minimum in point $(t^*,x^*)\in\Omega$ and function $\gamma^{\mathrm{T}}(t)x(t)$ is differentiable in this point, the following inequation is true:

$$\Psi(\gamma,x)= \\ =\gamma^{\mathrm{T}}(t)f(x)+\frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t)-\frac{1}{2}\gamma^{\mathrm{T}}(t)\Pi(x)\gamma(t)\le 0. \tag{5.2}$$

Definition 5.2. Lower viscosity solution of equation (5.1) is continuous function $\gamma^{\mathrm{T}}(t)x(t)$ meeting the following condition: if difference of functions $V(x)-\gamma^{\mathrm{T}}(t)x(t)$

reaches a local maximum in point $(t^*, x^*) \in \Omega$ and function $\gamma^{\mathrm{T}}(t)x(t)$ is differentiable in this point, the following inequation is true:

$$\Psi(\gamma, x) =$$
$$= \gamma^{\mathrm{T}}(t)f(x) + \frac{1}{2}x^{\mathrm{T}}(t)H^{\mathrm{T}}QHx(t) - \frac{1}{2}\gamma^{\mathrm{T}}(t)\Pi(x)\gamma(t) \geq 0. \qquad (5.3)$$

Definition 5.3. Viscosity solution is a function which is simultaneously upper and lower solution, i.e. meeting the following condition:

$$\gamma^{\mathrm{T}}(t)x(t) = V(x). \qquad (5.4)$$

To search for vector $\gamma(t)$ introduce Lyapunov function $V_L(x, \gamma)$

$$V_L(x, \gamma) = 0,5\Psi^2(x, \gamma). \qquad (5.5)$$

Total derivative of Lyapunov function (5.4) for stable system is

$$\frac{d}{dt}V_L(x, \gamma) = \qquad (5.6)$$
$$= \Psi(x, \gamma)\frac{\partial\Psi(x, \gamma)}{\partial x}\frac{dx(t)}{dt} + \Psi(x, \gamma)\frac{\partial\Psi(x, \gamma)}{\partial\gamma}\frac{d\gamma(t)}{dt} \leq 0.$$

Require that this condition be true when function $\gamma(t)$ providing fulfillment of "disappearing viscosity" condition [11] corresponds to solution of the following differential equation

$$\frac{d\gamma(t)}{dt} = -\left\{\frac{\partial\Psi(x, \gamma)}{\partial\gamma}\right\}^{\mathrm{T}}\Psi(x, \gamma),\ \gamma(t_0) = S(x_0)x_0. \qquad (5.7)$$

Taking into account that

$$\left\{\frac{\partial\Psi(x, \gamma)}{\partial\gamma}\right\}^{\mathrm{T}} = f(x) - \Pi(x)\gamma(t) = \frac{d}{dt}x(t)$$

rewrite equation (5.7):

$$\frac{d\gamma(t)}{dt} = -\Psi(x, \gamma)\frac{d}{dt}x(t),\ \gamma(t_0) = S(x_0)x_0. \qquad (5.8)$$

In view of condition (5.6), write condition of efficient operation of algorithm (5.7):

$$\left\|\frac{\partial\Psi(x, \gamma)}{\partial\gamma}\right\|^2\Psi^2(x, \gamma) \geq \left|\frac{\partial\Psi(x, \gamma)}{\partial x}\frac{dx(t)}{dt}\Psi(x, \gamma)\right|,\ x(t) \in \Omega_x. \qquad (5.9)$$

So, algorithm (5.7) in the problem of stabilizing of non-linear indefinite object may provide efficient operation of controls $w(t) = P^{-1}g_1^{\mathrm{T}}(x)\gamma(t)$, $u(t) = -R^{-1}g_2^{\mathrm{T}}(x)\gamma(t)$ when condition (5.9) is fulfilled.

Theorem 5.1. Algorithm (5.7) provides fulfillment of "vanishing viscosity" condition if

$$\left\|\frac{\partial\Psi(x, \gamma)}{\partial\gamma}\right\|^2\Psi^2(x, \gamma) \geq \left|\frac{\partial\Psi(x, \gamma)}{\partial x}\frac{dx(t)}{dt}\Psi(x, \gamma)\right|,\ x(t) \in \Omega_x.$$

6. CONCLUSIONS

The problem of optimal control for class of non-linear objects with uncontrolled restricted excitations is stated in the sense of a differential game. In case of problems with quadratic quality functional, the problem of search of optimal controls is reduced to finding of solution of partial derivatives Hamilton-Jacobi-Isaaks scalar equation. Solutions of this equation at the rate of functioning of the object are searched by means of special algorithmic procedures obtained with the use of theory of viscosity solution. The obtained results may be used for real-time solving of theoretical and applied problems of mathematics, mechanics, physics, biology, chemistry, engineering, control and navigation.

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